

Scaling invariant Harnack inequalities in a general setting

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Abstract

In a setting, where only “exit measures” are given, as they are associated with an arbitrary right continuous strong Markov process on a separable metric space, we provide simple criteria for the validity of Harnack inequalities for positive harmonic functions. These inequalities are scaling invariant with respect to a metric on the state space which, having an associated Green function, may be adapted to the special situation. In many cases, this also implies continuity of harmonic functions and Hölder continuity of bounded harmonic functions. The results apply to large classes of Lévy (and similar) processes.

Keywords: Harnack inequality; Hölder continuity; right process; balayage space; Lévy process; Green function; 3G-property; capacity; Krylov-Safonov estimate; Ikeda-Watanabe formula.

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1 Overview

The study of Harnack inequalities for positive functions which are harmonic with respect to rather general partial differential operators of second order, diffusions respectively, has a long history (see [17] and the references therein). Fairly recently, during the last 15 years, Harnack inequalities have been investigated for harmonic functions with respect to various classes of discontinuous Markov processes, integro-differential operators respectively (see [1, 2, 7, 9, 14, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29]).

The aim of this paper is to offer a very general analytic approach to scaling invariant Harnack inequalities for positive universally measurable functions on a separable metric space X which are harmonic with respect to given “harmonic measures” μ_x^U (not charging U), $x \in U$, U open in X (see (2.6)). For $V \subset U$ the corresponding measures are supposed to be compatible in a way which is obvious for exit distributions of right continuous strong Markov processes and harmonic measures on balayage spaces (see Examples 2.1). An additional ingredient we shall need is a “quasi-capacity” on X having suitable scaling properties such that an estimate of Krylov-Safonov type holds (see (3.3)).

Then a certain property (HJ) of the measures μ_x^U , which in Examples 2.1 trivially holds for diffusions and harmonic spaces, is necessary and sufficient for the validity of scaling invariant Harnack inequalities (Theorem 3.3). For Lévy processes it is

easy to specify simple properties of the Lévy measure which imply (HJ) for the exit distributions (see, for example, Lemma 5.1).

In Section 4, we discuss properties of an associated “Green function” which allow us to prove a Krylov-Safonov estimate for the corresponding capacity. This leads to Theorem 4.10 and, using recent results on Hölder continuity from [12], to Theorem 4.12 on (Hölder) continuity of harmonic functions.

After a first application to Lévy processes (Theorem 5.2) we discuss consequences of an estimate of Ikeda-Watanabe type (Theorems 6.2 and 6.3).

In a final Section 7, we indicate how a Green function satisfying (only) a weak 3G-property leads to Harnack inequalities which are scaling invariant with respect to an intrinsically defined metric.

2 Harmonic measures and harmonic functions

Let (X, ρ) be a separable metric space. In fact, the separability will only be used to ensure that finite measures μ on its σ -algebra $\mathcal{B}(X)$ of Borel subsets satisfy

$$(2.1) \quad \mu(A) = \sup\{\mu(F) : F \text{ closed}, F \subset A\}, \quad A \in \mathcal{B}(X)$$

(recall that every finite measure on the completion of X is tight).

For every open set Y in X , let $\mathcal{U}(Y)$ denote the set of all open sets U such that the closure \overline{U} of U is contained in Y . Given a set \mathcal{F} of numerical functions on X , let $\mathcal{F}_b, \mathcal{F}^+$ be the set of all functions in \mathcal{F} which are bounded, positive respectively. Let $\mathcal{M}(X)$ denote the set of all finite measures on $(X, \mathcal{B}(X))$ (which we also consider as measures on the σ -algebra $\mathcal{B}^*(X)$ of all universally measurable sets). For every $\mu \in \mathcal{M}(X)$, let $\|\mu\|$ denote the total mass $\mu(X)$.

For sufficient flexibility in applications, we consider harmonic measures only for open sets which are contained in a given open set X_0 of X . More precisely, we suppose that we have measures $\mu_x^U \in \mathcal{M}(X)$, $x \in X$, $U \in \mathcal{U}(X_0)$, such that the following hold for all $x \in X$ and $U, V \in \mathcal{U}(X_0)$ (where ε_x is the Dirac measure at x):

(M₀) *The measure μ_x^U is supported by U^c , and $\|\mu_x^U\| \leq 1$. If $x \in U^c$, then $\mu_x^U = \varepsilon_x$.*

(M₁) *The functions $y \mapsto \mu_y^U(E)$, $E \in \mathcal{B}(X)$, are universally measurable on X and*

$$(2.2) \quad \mu_x^U = (\mu_x^V)^U := \int \mu_y^U d\mu_x^V(y), \quad \text{if } V \subset U.$$

Of course, stochastic processes and potential theory abundantly provide examples (with $X_0 = X$).

EXAMPLES 2.1. 1. Right process \mathfrak{X} with strong Markov property on a Radon space X and

$$\mu_x^U(E) := \mathbb{P}^x[X_{\tau_U} \in E], \quad E \in \mathcal{B}(X),$$

where $\tau_U := \inf\{t \geq 0 : X_t \in U^c\}$ ([4, Propositions 1.6.5 and 1.7.11, Theorem 1.8.5]).

If $U, V \in \mathcal{U}(X_0)$ with $V \subset U$, then $\tau_U = \tau_V + \tau_U \circ \theta_{\tau_V}$, and hence, by the strong Markov property, for all $x \in X$ and $E \in \mathcal{B}(X)$,

$$\mu_x^U(E) = \mathbb{P}^x[X_{\tau_U} \in E] = \mathbb{E}^x(\mathbb{P}^{X_{\tau_V}}[X_{\tau_U} \in E]) = \int \mu_y^U(E) d\mu_x^V(y).$$

2. Balayage space (X, \mathcal{W}) (see [5, 10]) such that $1 \in \mathcal{W}$,

$$\int v d\mu_x^U = R_v^{U^c}(x) := \inf\{w(x) : w \in \mathcal{W}, w \geq v \text{ on } U^c\}, \quad v \in \mathcal{W}.$$

The properties (M_0) and (M_1) follow from [5, VI.2.1, 2.4, 2.10, 9.1].

Going back to the general setting, let us consider $x \in X$ and $U, V \in \mathcal{U}(X_0)$ such that $V \subset U$, and note first that, having (M_0) , equality (2.2) amounts to the equality

$$(2.3) \quad \mu_x^U = \mu_x^V|_{U^c} + \int_U \mu_y^U d\mu_x^V(y)$$

showing that $\mu_x^U \geq \mu_x^V$ on U^c . However,

$$(2.4) \quad \|\mu_x^U\| = \|(\mu_x^V)^U\| = \int \|\mu_y^U\| d\mu_x^V(y) \leq \|\mu_x^V\|.$$

In particular, for any $A \in \mathcal{B}(X)$ containing U ,

$$(2.5) \quad \mu_x^U(A^c) \geq \mu_x^V(A^c) \quad \text{and} \quad \mu_x^U(A) \leq \mu_x^V(A).$$

For every $U \in \mathcal{U}(X_0)$, let $\mathcal{H}(U)$ denote the set of all universally measurable real functions h on X which are *harmonic on U* , that is, such that, for all $V \in \mathcal{U}(U)$ and $x \in V$, the function h is μ_x^V -integrable and

$$(2.6) \quad \int h d\mu_x^V = h(x).$$

It is easily seen that, for all bounded universally measurable functions f on X and $U \in \mathcal{U}(X_0)$, the function

$$(2.7) \quad h: x \mapsto \int f d\mu_x^U$$

is harmonic on U . Indeed, it suffices to consider the case $f = 1_{E_0}$, $E_0 \in \mathcal{B}^*(X)$. Let us fix $U \in \mathcal{U}(X_0)$, $V \in \mathcal{U}(U)$ and $x \in X$. Then there are $E_1, E_2 \in \mathcal{B}(X)$ such that $E_1 \subset E_0 \subset E_2$ and $\mu_x^U(E_1) = \mu_x^U(E_2)$. By (2.2),

$$(2.8) \quad \mu_x^U(E_j) = \int \mu_y^U(E_j) d\mu_x^V(y) \quad \text{for } j = 1, 2.$$

Hence $\mu_y^U(E_1) = \mu_y^U(E_2)$ for μ_x^V -a.e. $y \in X$. This implies that the equality (2.8) holds as well for $j = 0$.

3 Scaling invariant Harnack inequalities

Let us define

$$U(x, r) := \{y \in X : \rho(x, y) < r\}, \quad x \in X, r > 0.$$

Moreover, let $R_0(x) := \sup\{r > 0 : \overline{U(x, r)} \subset X_0\}$, $x \in X_0$, and

$$\mathcal{U}_0 := \{U(x, r) : x \in X_0, r < R_0(x)\}.$$

We are interested in the following Harnack inequalities:

(HI) There exist $\alpha \in (0, 1)$ and $K \geq 1$ such that, for all $U(x_0, R) \in \mathcal{U}_0$,

$$\sup h(U(x_0, \alpha R)) \leq K \inf h(U(x_0, \alpha R)) \quad \text{for all } h \in \mathcal{H}_b^+(U(x_0, R)).$$

Let us immediately note consequences for arbitrary positive harmonic functions.

PROPOSITION 3.1. *Suppose that (HI) holds with $\alpha \in (0, 1)$ and $K \geq 1$.*

1. *Then, for all $U(x_0, R) \in \mathcal{U}_0$ and $h \in \mathcal{H}^+(U(x_0, R))$,*

$$(3.1) \quad \sup h(U(x_0, \alpha R)) \leq K \inf h(U(x_0, \alpha R)).$$

2. *If, for all $U \in \mathcal{U}(X_0)$, the functions in $\mathcal{H}_b^+(U)$ are continuous on U , then, for all $U \in \mathcal{U}(X_0)$, the functions in $\mathcal{H}^+(U)$ are continuous on U .*

Proof. Let $U(x_0, R) \in \mathcal{U}_0$ and $V_r := U(x_0, r)$, $0 < r \leq R$. Further, let $h \in \mathcal{H}^+(V_R)$ and $0 < R' < R$.

1. By (2.7), we may define functions $h_n \in \mathcal{H}_b^+(V_{R'})$, $n \in \mathbb{N}$, by

$$(3.2) \quad h_n(x) := \int (h \wedge n) d\mu_x^{V_{R'}}, \quad x \in X.$$

Then $\sup h_n(V_{\alpha R'}) \leq K \inf h_n(V_{\alpha R'})$ for every $n \in \mathbb{N}$. Clearly, $h_n \uparrow h$ as $n \rightarrow \infty$. Thus $\sup h(V_{\alpha R'}) \leq K \inf h(V_{\alpha R'})$. Since $R' \in (0, R)$ was arbitrary, (3.1) follows.

2. Since $h - h_n \in \mathcal{H}^+(V_{R'})$ for every $n \in \mathbb{N}$, we now see, by (3.1), that

$$h - h_n \leq K(h - h_n)(x_0) \quad \text{on } V_{\alpha R'}.$$

So $h_n \rightarrow h$ uniformly on $V_{\alpha R'}$. Therefore h is continuous on $V_{\alpha R'}$. \square

Given $c \geq 1$, an increasing positive function $A \mapsto m(A)$, A universally measurable in X_0 , will be called a *quasi-capacity* with constant $c \geq 1$ on X_0 if, for all universally measurable sets A, B in X_0 ,

$$m(A) = \sup\{m(F) : F \subset A, F \text{ closed}\} \quad \text{and} \quad m(A \cup B) \leq c(m(A) + m(B)).$$

Clearly, every $\mu \in \mathcal{M}(X)$ (restricted to X_0) is a quasi-capacity on X_0 , and we note already now that the capacity set function cap which will be defined in Section 4 is a quasi-capacity (both with constant 1).

To obtain suitable criteria for the validity of (HI), we suppose that we have a quasi-capacity m on X_0 , an increasing continuous function $m_0 : (0, \infty) \rightarrow (0, \infty)$ and $\alpha, a, \eta \in (0, 1/3)$, $c_0 \geq 1$ such that the following translation property (T), scaling property (SC) and estimate (KS) of Krylov-Safonov type hold:

$$(T) \quad \text{For every } U(x, r) \in \mathcal{U}_0, \quad c_0^{-1} m_0(r) \leq m(U(x, r)) \leq c_0 m_0(r).$$

$$(SC) \quad \text{For every } r > 0, \quad a m_0(r) \leq m_0(\alpha r), \quad \text{and } \lim_{r \rightarrow 0} m_0(r) = 0.$$

$$(KS) \quad \text{For all } U(x, r) \in \mathcal{U}_0, \quad y \in U(x, \alpha r) \quad \text{and closed sets } F \subset U(x, \alpha r),$$

$$(3.3) \quad \mu_y^{U(x, r) \setminus F}(F) \geq \eta \frac{m(F)}{m(U(x, r))}.$$

REMARK 3.2. Let us observe that (KS) is much weaker than the property of Krylov-Safonov type we may prove under rather general assumptions on an associated Green function, which yield that (3.3) holds with $m(U(x, \alpha r))$ in place of $m(U(x, r))$ (see Proposition 4.9).

Finally, let us consider the following property which, of course, trivially holds if the measures $\mu_x^{U(x, \alpha r)}$ are supported by the boundary of $U(x, \alpha r)$.

(HJ) *There exist $\alpha \in (0, 1)$, $c_J \geq 1$ such that, for all $x \in X_0$ and $0 < r < \alpha R_0(x)$,*

$$(3.4) \quad \mu_x^{U(x, \alpha r)} \leq c_J \mu_y^{U(x, r)} \text{ on } U(x, r)^c, \quad y \in U(x, \alpha^2 r).$$

Clearly, (HI) implies (HJ), since, for every $E \in \mathcal{B}(X)$ with $E \subset U(x, r)^c$, the function $z \mapsto \mu_z^{U(x, r)}(E)$ is harmonic on $U(x, r)$ (see (2.7)) and $\mu_y^{U(x, \alpha r)}(E) \leq \mu_y^{U(x, r)}(E)$, by (2.3). Moreover, again by (2.3), if (3.4) holds for some $\alpha \in (0, 1)$, then it holds for every smaller α . By the same argument, (HJ) is equivalent to the following property (at the expense of replacing α by $\alpha/2$).

(HJ*) *There exist $\alpha \in (0, 1)$, $c_J \geq 1$ such that, for all $x \in X_0$ and $0 < r < \alpha R_0(x)$,*

$$(3.5) \quad \mu_x^{U(x, \alpha r)} \leq c_J \mu_y^{U(y, r)} \text{ on } U(x, r)^c, \quad y \in U(x, \alpha^2 r).$$

Indeed, suppose that (HJ*) is satisfied. Let $x \in X_0$, $0 < r < (\alpha/2)R_0(x)$ and $y \in U(x, (\alpha/2)^2 r)$. Then $U(y, r/2) \subset U(x, r)$, and hence

$$\mu_x^{U(x, \alpha r/2)} \leq \mu_y^{U(y, r/2)} \leq \mu_y^{U(x, r)} \quad \text{on } U(x, r)^c.$$

So (HJ) holds. Similarly for the reverse implication.

Our main result on scaling invariant Harnack inequalities (assuming the properties (T), (SC) and (KS)) is as follows.

THEOREM 3.3. *(HI) holds if and only if (HJ) holds. In particular, (HI) holds if the measures $\mu_x^{U(x, r)}$, $U(x, r) \in \mathcal{U}_0$, are supported by the boundary of $U(x, r)$.*

To prove this result let us assume for the remainder of this section that (HJ) holds. We shall modify our results from [14] (which were inspired by [1]). Suppose that (SC) and (KS) hold with a_1, α_1 and (HJ) holds with α_2 .

We choose first

$$(3.6) \quad 0 < \alpha \leq (\alpha_1 \wedge \alpha_2)/4,$$

next $l \in \mathbb{N}$ with $\alpha_1^l \leq \alpha$, and define $a := a_1^l$. Then (SC), (KS) and (HJ) (see (2.3)) hold with these α, a . Let

$$\beta := \frac{\eta a}{4cc_0^2} \quad \text{and} \quad \tilde{\beta} := \frac{\beta}{c_J}.$$

Of course, $\tilde{\beta} \leq \beta \leq 1/4$. We choose $j_0, k_0 \in \mathbb{N}$ such that

$$(3.7) \quad a(1 + \beta)^{j_0} > 1, \quad \alpha^{k_0-1} < (1 - \alpha)/j_0,$$

and fix

$$(3.8) \quad K \geq \frac{2cc_0^2(1+\beta)}{\eta\tilde{\beta}a^{k_0+2}}.$$

Let us now fix $x \in X_0$ and $0 < R < R_0(x)$, take $r_0 := \alpha^{k_0}R$ and choose $r_n \in (0, r_0)$ with

$$(3.9) \quad m_0(r_n) = (1+\beta)^{-n}m_0(r_0), \quad n \in \mathbb{N}.$$

We claim that

$$(3.10) \quad \sum_{n \in \mathbb{N}} r_n < \alpha R.$$

Indeed, if $n = i + kj_0$, where $1 \leq i \leq j_0$ and $k \geq 0$, then, by (3.9), (3.7) and (SC),

$$m_0(r_n) < (1+\beta)^{-kj_0}m_0(r_0) \leq a^k m_0(r_0) \leq m_0(\alpha^k r_0),$$

and hence $r_n < \alpha^k r_0$. So, by (3.7),

$$\sum_{n \in \mathbb{N}} r_n < j_0 \sum_{k=0}^{\infty} \alpha^k r_0 < \alpha R.$$

In connection with (3.10) the following result will immediately yield Theorem 3.3.

PROPOSITION 3.4. *Let $h \in \mathcal{H}_b^+(U(x_0, R))$ and suppose that $n \in \mathbb{N}$ and there exists $x \in U(x_0, 2\alpha R)$ with*

$$(3.11) \quad h(x) > (1+\beta)^{n-1}K \inf h(U(x_0, \alpha R)).$$

Then there exists a point $x' \in U(x, r_n)$ such that

$$h(x') > (1+\beta)h(x) > (1+\beta)^n K \inf h(U(x_0, \alpha R)).$$

Proof. By (3.11), there is a point $y_0 \in U(x_0, \alpha R)$ such that $h(y_0) > (1+\beta)^{n-1}Kh(y_0)$. Replacing h by $h/h(y_0)$, if $h(y_0) > 0$, we may assume that $h(y_0) \leq 1$.

For every $0 < s < R$, let $U_s := U(x_0, s)$. Moreover, let $r := r_n$, $R' := (3/4)R$ and

$$B := U(x, r), \quad B' := U(x, \alpha r), \quad B'' := U(x, \alpha^2 r), \quad A := \{y \in B'' : h(y) \geq \tilde{\beta}h(x)\}.$$

Then $B \subset U_{4\alpha R'} \subset U_{\alpha_2 R'}$ and $y_0 \in U_{\alpha_2 R'}$. By (KS), for every closed $E \subset A$,

$$1 \geq h(y_0) = \mu_{y_0}^{U_{R'} \setminus E}(h) \geq \tilde{\beta}h(x)\mu_{y_0}^{U_{R'} \setminus E}(E) \geq \tilde{\beta}h(x)\eta m(E)/m(U_{R'}),$$

and hence $m(A) < (\eta\tilde{\beta}(1+\beta)^{n-1}K)^{-1}m(U_R)$. By (T), (SC) and (3.9),

$$m(U_R) \leq c_0 m_0(R) \leq c_0 a^{-k_0} m_0(r_0) = c_0 a^{-k_0} (1+\beta)^n m_0(r).$$

By (3.8), we therefore conclude that

$$(3.12) \quad 2cm(A) < \frac{2cm(U_R)}{\eta\tilde{\beta}(1+\beta)^{n-1}K} \leq c_0^{-1}a^2 m_0(r) \leq c_0^{-1}m_0(\alpha^2 r) \leq m(B'').$$

Since $m(B'') \leq c(m(A) + m(B'' \setminus A))$, we obtain that $m(B'' \setminus A) > (2c)^{-1}m(B'')$, where $m(B'') \geq ac_0^{-2}m(B')$, by (T) and (SC). So there is a closed set F in $B'' \setminus A$ with

$$m(F) > a(2cc_0^2)^{-1}m(B').$$

Let

$$\nu := \mu_x^{B' \setminus F}.$$

The measure ν is supported by $F \cup (X \setminus B')$. By (KS), we see that

$$(3.13) \quad \nu(F) \geq \eta m(F)/m(B') > \eta a(2cc_0^2)^{-1} = 2\beta.$$

Moreover,

$$(3.14) \quad h(x) = \int h d\nu \quad \text{and} \quad \int_F h d\nu \leq \tilde{\beta}h(x)\nu(F) < \beta h(x)\nu(F).$$

In particular, ν is not supported by F .

Next we claim that the function $H := 1_{B^c}h$ satisfies $\mu_x^{B'}(H) \leq \beta h(x)$. Indeed, if not, then (HJ) implies that, for every $y \in B''$,

$$h(y) = \mu_y^B(h) = \mu_y^B(H) \geq c_J^{-1}\mu_x^{B'}(H) > c_J^{-1}\beta h(x) = \tilde{\beta}h(x),$$

contradicting the fact that $A \neq B''$, by (3.12). Therefore, by (2.3),

$$\int_{B^c} h d\nu = \nu(H) \leq \mu_x^{B'}(H) \leq \beta h(x).$$

Defining $b := \sup h(B)$ we have

$$\int_{B \setminus B'} h d\nu \leq b\nu(B \setminus B') \leq b(1 - \nu(F))$$

(where we used that $\|\nu\| \leq 1$). Hence, by (3.14),

$$h(x) = \int_F h d\nu + \int_{X \setminus B'} h d\nu \leq \beta h(x)\nu(F) + \beta h(x) + b(1 - \nu(F)).$$

Since $\nu(F) > 2\beta$, by (3.13), we thus conclude that

$$b \geq \frac{1 - \beta - \beta\nu(F)}{1 - \nu(F)} h(x) > (1 + \beta)h(x) > (1 + \beta)^n K$$

completing the proof (we have $(1 + \beta)(1 - \nu(F)) = 1 + \beta - \nu(F) - \beta\nu(F)$). \square

Proof of Theorem 3.3. Let $h \in \mathcal{H}_b^+(U(x_0, R))$ and $\gamma := \inf h(U(x_0, \alpha r))$. If

$$\sup h(U(x_0, \alpha r)) \leq K\gamma$$

does not hold, then there exists a point $x_1 \in U(x_0, \alpha R)$ such that $h(x_1) > K\gamma$. Proposition 3.4 and (3.10) then recursively lead to points x_2, x_3, \dots in $U(x_0, 2\alpha R)$ satisfying $h(x_n) > (1 + \beta)^{n-1}h(x)$ for every $n \geq 2$. This contradicts the boundedness of h . \square

4 Green function and Harnack inequalities

In this section, we assume that for *all* open sets U in the separable metric space (X, ρ) , we have measures $\mu_x^U \in \mathcal{M}(X)$, $x \in X$, such that (M_0) and (M_1) hold (see Section 2).

Let $G: X \times X \rightarrow (0, \infty]$ be a Borel measurable function and, for $\mu \in \mathcal{M}(X)$, let

$$G\mu(x) := \int G(x, y) d\mu(y), \quad x \in X.$$

For every $A \in \mathcal{B}^*(X)$, let

$$\text{cap } A := \sup\{\|\mu\| : \mu \in \mathcal{M}(X), \mu(A^c) = 0, G\mu \leq 1\}.$$

Clearly,

$$\text{cap } A = \sup\{\text{cap } F : F \text{ closed set, } F \subset A\},$$

and the mapping $A \mapsto \text{cap } A$ is increasing and subadditive. So, in our terminology, $m := \text{cap}$ is a quasi-capacity (with constant 1) on X .

As in Section 2, we suppose that X_0 is an open set in X and $\mathcal{U}(X_0)$ denotes the set of all open sets U with $\overline{U} \subset X_0$. For every closed set A in X , let

$$\varepsilon_x^A := \mu_x^{X \setminus A}.$$

REMARK 4.1. For a right process \mathfrak{X} (see Example 2.1,1), ε_x^A is given by

$$\varepsilon_x^A(E) = \mathbb{P}^x[X_{D_A} \in E], \quad E \in \mathcal{B}(X),$$

where $D_A := \inf\{t \geq 0 : X_t \in A\}$ denotes the time of the first entry into A . For a balayage space (X, \mathcal{W}) with $1 \in \mathcal{W}$ (see Example 2.1,2), the measure ε_x^A is the reduced measure for x and A (see [5, VI, p. 67]).

Let us consider the following properties.

(G₁) *There exists $c_1 \geq 1$ such that, for all $U \in \mathcal{U}(X_0)$, $x \in U$, $\delta > 0$, the following holds: For every closed set $A \subset U$, there exists a closed neighborhood $B \subset U$ of A and a measure ν on B such that*

$$(4.1) \quad \|\varepsilon_x^B\| - \delta < c_1 \|\varepsilon_x^A\| \quad \text{and} \quad \|\varepsilon_y^A\| \leq G\nu(y) \leq c_1 \|\varepsilon_y^B\|, \quad y \in X.$$

(G₂) *There are a strictly decreasing continuous function $g: [0, \infty) \rightarrow (0, \infty]$ and constants $c, c_D, M_0 \in [1, \infty)$, $\alpha_0 \in (0, 1)$ such that, for every $r > 0$,*

$$g(r/2) \leq c_D g(r), \quad M_0 g(r) \leq g(\alpha_0 r) \quad \text{and} \quad c^{-1} g \circ \rho \leq G \leq c g \circ \rho.$$

(G₃) *There exists $c_2 \geq 1$ such that, for $x \in X_0$ and $0 < r < R_0(x)$,*

$$(4.2) \quad \text{cap } U(x, r) \geq c_2^{-1} g(r)^{-1}.$$

Let us first note that, having (G₂), for every $M \geq 1$, there exists $0 < \alpha_M < 1/4$ such that

$$(4.3) \quad M g(r) \leq g(\alpha_M r) \quad \text{for every } r > 0.$$

Indeed, it suffices to choose $k \in \mathbb{N}$ with $M \leq M_0^k$, $\alpha_0^k < 1/4$, and to take $\alpha_M := \alpha_0^k$.

REMARK 4.2. Of course, (G_1) holds in Example 2.1,2, if the functions $G(\cdot, y)$ are potentials on X with superharmonic support $\{y\}$ and, for every continuous real potential p on X , there exists a measure ν on X such that $p = G\nu$ (cf. [13]). Indeed, let A be a closed set, $A \subset U \in \mathcal{U}(X_0)$ and $\delta > 0$. Then, by [5, VI.1.2], there exists a closed neighborhood $B \subset U$ of A such that $\|\varepsilon_x^B\| - \delta < \|\varepsilon_x^A\|$. Let $\varphi \in \mathcal{C}(X)$, $1_A \leq \varphi \leq 1_B$ and

$$p := R_\varphi := \inf\{w \in \mathcal{W} : w \geq \varphi\}.$$

Then p is a continuous real potential which is harmonic on $X \setminus B$. Hence there exists a measure ν , which is supported by B , such that $p = G\nu$, and we have $\|\varepsilon_y^A\| \leq G\nu(y) \leq \|\varepsilon_y^B\|$ for every $y \in X$.

PROPOSITION 4.3. *Property (G_2) implies the following.*

- For every $x \in X$, $G(x, x) = \lim_{y \rightarrow x} G(y, x) = \infty$.
- The function G has the triangle property: There exists $C \geq 1$, such that

$$(4.4) \quad G(x, z) \wedge G(y, z) \leq CG(x, y), \quad x, y, z \in X.$$

- For all $x \in X$ and neighborhoods V of x , $G(\cdot, x)$ is bounded on V^c .

Proof. By (4.3), $\lim_{r \rightarrow 0} g(r) = \infty$. Hence the inequality $c^{-1}g \circ \rho \leq G$ implies that $G(x, x) = \lim_{y \rightarrow x} G(y, x) = \infty$. Moreover, if $x, y, z \in X$, then $\rho(x, z) \geq \rho(x, y)/2$ or $\rho(z, y) \geq \rho(x, y)/2$. So $G(x, z) \leq c^2 c_D G(x, y)$ or $G(y, z) \leq c^2 c_D G(x, y)$. Finally, the last property is satisfied, since $G(y, x) \leq cg(r)$ if $\rho(y, x) \geq r$. \square

REMARK 4.4. In Section 7, we shall see that, conversely, the properties of G stated in Proposition 4.3 enable the construction of a metric $\tilde{\rho}$ satisfying (G_2) with $\tilde{g}(r) = r^{-\gamma}$ for some $\gamma \geq 1$.

LEMMA 4.5. 1. For every $r > 0$, $\text{cap} U(x, r) \leq cg(r)^{-1}$.

2. If $A \in \mathcal{B}^*(X)$, $\nu \in \mathcal{M}(X)$ and $G\nu \geq 1$ on A , then $\text{cap} A \leq c^2 \|\nu\|$.

Proof. Let $\mu \in \mathcal{M}(X)$, $G\mu \leq 1$. If $\mu(U(x, r)^c) = 0$, then

$$c^{-1}g(r)\|\mu\| \leq \int G(x, y) d\mu(y) = G\mu(x) \leq 1.$$

If $A \in \mathcal{B}^*(X)$, $\mu(A^c) = 0$ and $\nu \in \mathcal{M}(X)$ with $G\nu \geq 1$ on A , then

$$\|\mu\| \leq \int G\nu d\mu \leq c^2 \int G\mu d\nu \leq c^2 \|\nu\|.$$

\square

Let us introduce the following property.

(\overline{G}_3) There exists $c_2 \geq 1$ such that, for $x \in X_0$ and $0 < r < R_0(x)$,

$$(4.5) \quad \|\varepsilon_y^{\overline{U(x, r)}}\| \geq c_2^{-1}g(r)^{-1}G(y, x), \quad y \in U(x, r)^c.$$

The next proposition is of independent interest, since assuming that we have a \mathcal{P} -harmonic space where 1 is superharmonic (that is, a balayage space (X, \mathcal{W}) with $1 \in \mathcal{W}$, where the harmonic measures μ_x^U are supported by the boundary of U), and $G(\cdot, x)$ is a potential which is harmonic on the complement of $\{x\}$, we trivially have

$$\|\varepsilon_y^{\overline{U(x,r)}}\| \geq c^{-1}g(r)^{-1}G(y, x), \quad y \in U(x, r)^c.$$

Its second part will be used in Section 7.

PROPOSITION 4.6. *Suppose that (G_1) and (G_2) hold.*

1. *Property (G_3) implies (\overline{G}_3) .*
2. *Suppose that $X_0 \neq X$ and (\overline{G}_3) holds. Then (G_3) holds.*

Proof. Let $x \in X_0$, $0 < r < R_0(x)$, $z \in U(x, r)^c$ (such a point exists if $X_0 \neq X$). By (G_1) , there exists a closed neighborhood B of $A := \overline{U(x, r/4)}$ in $U(x, r/2)$ and a measure ν on B such that

$$\|\varepsilon_y^A\| \leq G\nu(y) \leq c_1 \|\varepsilon_y^B\| \quad \text{for every } y \in X.$$

Then $\|\nu\| \leq c_1 \text{cap } U(x, r)$ and, by Lemma 4.5,2, $\text{cap } A \leq c^2 \|\nu\|$. Let $s := \rho(x, z)$. Obviously, $s/2 \leq \rho(z, \cdot) \leq 2s$ on B , and hence

$$cg(s/2)\|\nu\| \geq G\nu(z) \geq c^{-1}g(2s)\|\nu\| \geq (c^2c_D)^{-1}G(z, x)\|\nu\|.$$

1. Assuming (G_3) , clearly $\text{cap } A \geq \text{cap } U(x, r/4) \geq c_2^{-1}g(r/4)^{-1} \geq (c_D^2c_2)^{-1}g(r)^{-1}$, and therefore, using also (2.4),

$$c_1 \|\varepsilon_z^{\overline{U(x,r)}}\| \geq c_1 \|\varepsilon_z^B\| \geq G\nu(z) \geq (c^2c_D)^{-1}G(z, x)\|\nu\| \geq (c^4c_D^3c_2)^{-1}g(r)^{-1}G(z, x).$$

2. Assuming (\overline{G}_3) , we have $\|\varepsilon_z^A\| \geq c_2^{-1}g(r/4)^{-1}G(z, x) \geq (cc_D^2c_2)^{-1}g(r)^{-1}g(s)$, where

$$\|\varepsilon_z^A\| \leq G\nu(z) \leq cg(s/2)\|\nu\| \leq cc_Dc_1g(s) \text{cap } U(x, r),$$

and hence $\text{cap } U(x, r) \geq (c^2c_D^3c_1c_2)^{-1}g(r)^{-1}$. \square

REMARK 4.7. The proof of (2) shows that (G_3) already holds if, given $x \in X_0$ and $0 < r < R_0(x)/4$, (4.5) is satisfied for just one point $y \in U(x, 4r)^c$.

From now on let us assume in this section that (G_1) , (G_2) , (G_3) hold,

$$(4.6) \quad M := (2c_D^2c^2c_1^2) \vee (3cc_2) \quad \text{and} \quad 0 < \alpha \leq \alpha_M$$

so that $Mg(r) \leq g(\alpha r)$ for every $r > 0$. We intend to prove that we have the properties (T), (SC) and (KS) taking $m(A) := \text{cap } A$ and $m_0(r) := g(r)^{-1}$. Of course, (T) follows from Lemma 4.5,1 and (G_3) . If $k \in \mathbb{N}$ such that $2^{-k} \leq \alpha$, then (SC) holds with $a := c_D^{-k}$ by the doubling property of g . To get (KS) we first note the following.

LEMMA 4.8. *Let U be an open set in X and $A \subset U$ be a closed set. Then*

$$\varepsilon_x^{A \cup U^c}(U) \geq \|\varepsilon_x^A\| - \sup_{z \in U^c} \|\varepsilon_z^A\| \quad \text{for every } x \in X.$$

Proof. Since $\varepsilon_x^{A \cup U^c}$ is supported by $A \cup U^c$, we know, by (2.2), that

$$\|\varepsilon_x^A\| = \int_{A \cup U^c} \|\varepsilon_z^A\| d\varepsilon_x^{A \cup U^c}(z).$$

The proof is completed observing that $\varepsilon_z^A = \varepsilon_z$, $z \in A$, and $\varepsilon_x^{A \cup U^c}(U^c) \leq 1$. \square

The next proposition establishes (KS) even with $m(U(x, \alpha r))$ in place of $m(U(x, r))$ in (3.3) (where we may remember that, in Example 2.1,1, we have $\varepsilon_x^{A \cup U(x_0, r)^c}(A) = \mathbb{P}^x[D_A < \tau_{U(x_0, r)}]$).

PROPOSITION 4.9. *Let $\eta := (2c_D c^3 c_1^2 c_2)^{-1}$, $U(x_0, r) \in \mathcal{U}_0$, $x \in U(x_0, \alpha r)$ and $A \subset U(x_0, \alpha r)$ be closed. Then*

$$(4.7) \quad \varepsilon_x^{A \cup U(x_0, r)^c}(A) \geq \eta \frac{\text{cap } A}{\text{cap } U(x_0, \alpha r)}.$$

Proof. Let $\delta > 0$ and $z \in U(x_0, r)^c$. By (G₁), there are a closed neighborhood B of A , $B \subset U(x_0, \alpha r)$, and a measure ν on B such that

$$\|\varepsilon_x^B\| < c_1(\|\varepsilon_x^A\| + \delta) \quad \text{and} \quad \|\varepsilon_y^A\| \leq G\nu(y) \leq c_1\|\varepsilon_y^B\| \quad \text{for every } y \in X.$$

Since $\rho(x, \cdot) \leq 2\alpha r$ on B , we obtain that

$$c_1^{-1}\|\varepsilon_x^B\| \geq c_1^{-2} \int G(x, \cdot) d\nu \geq (cc_1^2)^{-1}g(2\alpha r)\|\nu\| \geq (c_D cc_1^2)^{-1}g(\alpha r)\|\nu\|.$$

Further, $\rho(z, \cdot) \geq r/2$ on B and $cg(r/2) \leq c_D cg(r) \leq (2c_D cc_1^2)^{-1}g(\alpha r)$, by (4.6). So

$$\|\varepsilon_z^A\| \leq \int G(z, \cdot) d\nu \leq cg(r/2)\|\nu\| \leq (2c_D cc_1^2)^{-1}g(\alpha r)\|\nu\|.$$

Combining these two estimates we see that

$$\|\varepsilon_x^A\| + \delta - \|\varepsilon_z^A\| \geq c_1^{-1}\|\varepsilon_x^B\| - \|\varepsilon_z^A\| \geq (2c_D cc_1^2)^{-1}g(\alpha r)\|\nu\|.$$

Therefore, by Lemma 4.8, Lemma 4.5,2 and (4.2),

$$\varepsilon_x^{A \cup U(x_0, r)^c}(U(x_0, r)) + \delta > (2c_D cc_1^2)^{-1}g(\alpha r)\|\nu\| \geq \eta \text{cap } A / \text{cap } U(x_0, \alpha r).$$

Since the measure $\varepsilon_x^{A \cup U(x_0, r)^c}$ is supported by $A \cup U(x_0, r)^c$, the proof is finished. \square

By Theorem 3.3, we now obtain the following result.

THEOREM 4.10. *Suppose that we have (HJ) and $\alpha \in (0, 1)$ which satisfies (3.6) and (4.6). Then there exists $K \geq 1$ such that, for all $U(x, R) \in \mathcal{U}_0$,*

$$\sup h(U(x, \alpha R)) \leq K \inf h(U(x, \alpha R)), \quad h \in \mathcal{H}^+(U(x, R)).$$

We recall that, by Proposition 3.1, (HI) and the continuity of all functions in $\mathcal{H}_b^+(U)$, $U \in \mathcal{U}(X_0)$, imply the continuity of all functions in $\mathcal{H}^+(U)$, $U \in \mathcal{U}(X_0)$. In fact, assuming that the constant function 1 is harmonic on X , [12, Corollary 3.2] implies even the Hölder continuity of all functions in $\mathcal{H}_b^+(U)$, $U \in \mathcal{U}(X_0)$. To see this we only have to verify property (J₀) in [12], that is, the following.

PROPOSITION 4.11. *There exists $\delta_0 > 0$ such that, for every $U(x, r) \in \mathcal{U}_0$,*

$$(4.8) \quad \mu_x^{U(x, \alpha^2 r)}(U(x, r)) > \delta_0.$$

Proof. Let $U(x, r) \in \mathcal{U}_0$ and $S := U(x, \alpha r) \setminus U(x, \alpha^2 r)$. Then, by (4.2), Lemma 4.5,1 and (4.6),

$$\begin{aligned} \text{cap } S &\geq \text{cap } U(x, \alpha r) - \text{cap } U(x, \alpha^2 r) \\ &\geq c_2^{-1} g(\alpha r)^{-1} - c g(\alpha^2 r)^{-1} > (2c_2)^{-1} g(\alpha r)^{-1}. \end{aligned}$$

So $\text{cap } F > (2c_2)^{-1} g(\alpha r)^{-1}$ for some closed set $F \subset S$. By Proposition 4.9,

$$\mu_x^{U(x, r) \setminus F}(U(x, r)) = \varepsilon_x^{F \cup U(x, r)^c}(U(x, r)) \geq \eta \text{cap } F / \text{cap } U(x, \alpha r) \geq (2cc_2^2)^{-1} \eta.$$

To finish the proof we note that $\mu_x^{U(x, \alpha^2 r)}(U(x, r)) \geq \mu_x^{U(x, r) \setminus F}(U(x, r))$, by (2.5). \square

As already indicated, [12, Corollary 3.2] now yields the following result.

THEOREM 4.12. *Suppose that (HJ) holds and $1 \in \mathcal{H}(X)$. Then there exist $\beta \in (0, 1)$ and $C \geq 1$ such that, for all $U(x, R) \in \mathcal{U}_0$,*

$$|h(y) - h(x)| \leq C \|h\|_\infty \left(\frac{\rho(x, y)}{R} \right)^\beta \quad \text{for all } y \in U(x, R), h \in \mathcal{H}_b(U(x, R)).$$

In particular, every universally measurable function, which is harmonic on an open set U in X_0 , is continuous on U .

5 A first application to Lévy processes

In this section, let us assume that $X = \mathbb{R}^d$, $d \geq 1$, $\rho(x) = |x - y|$, and the measures μ_x^U are given by a Lévy process \mathfrak{X} on \mathbb{R}^d such that, for some Borel measurable function $n \geq 0$ on \mathbb{R}^d and all $x \in \mathbb{R}^d$, $r > 0$ and Borel sets A in $\mathbb{R}^d \setminus \overline{U(x, r)}$,

$$(5.1) \quad \mathbb{P}^x[X_{\tau_{U(x, r)}} \in A] = \mathbb{E}^x \int_0^{\tau_{U(x, r)}} \int_A n(z - X_u) dz du.$$

LEMMA 5.1. *Suppose that $0 < \alpha < 1/2$ and $c_J \geq 1$ such that, for $y, z \in \mathbb{R}^d$,*

$$(5.2) \quad n(z) \leq c_J n(z + y) \quad \text{provided } |y| < 2\alpha|z|.$$

Then for all $x \in \mathbb{R}^d$, $r > 0$ and $y \in U(x, \alpha r)$,

$$\mu_x^{U(x, \alpha r)} \leq c_J \mu_y^{U(y, \alpha r)} \quad \text{on } U(x, r)^c.$$

In particular, (HJ) holds.

Proof. By translation invariance, it suffices to consider the case $x = 0$. Let $r > 0$ and $y \in U(0, \alpha r)$, $\tau := \tau_{U(0, \alpha r)}$, and let A be a Borel set in $U(0, r)^c$. By (5.1) and translation invariance,

$$\begin{aligned} \mu_y^{U(y, \alpha r)}(A) &= \mu_0^{U(0, \alpha r)}(A - y) \\ &= \mathbb{E}^0 \int_0^\tau \int_{A-y} n(z - X_u) dz du = \mathbb{E}^0 \int_0^\tau \int_A n(z - X_u + y) dz du. \end{aligned}$$

If $z \in A$ and $\tilde{z} \in U(0, \alpha r)$, then $|y| < \alpha r < 2\alpha(1 - \alpha)r < 2\alpha|z - \tilde{z}|$, and hence $n(z - \tilde{z}) \leq c_J n(z - \tilde{z} + y)$, by (5.2). Considering also the case $y = 0$, we conclude that $\mu_0^{U(0, \alpha r)}(A) \leq c_J \mu_y^{U(y, \alpha r)}(A)$. To complete the proof it suffices to recall that $\mu_y^{U(y, \alpha r)} \leq \mu_y^{U(0, r)}$ on $U(0, r)^c$, by (2.3). \square

So, by Theorems 4.10 and 4.12, we have the following.

THEOREM 5.2. *Suppose that n satisfies (5.2) and that there exists a Borel measurable function $G: \mathbb{R}^d \times \mathbb{R}^d \rightarrow (0, \infty]$ satisfying $(G_1) - (G_3)$ with $\rho(x, y) = |x - y|$. Then there exist $\alpha \in (0, 1)$ and $K \geq 1$ such that, for all $x \in \mathbb{R}^d$ and $R > 0$,*

$$\sup h(U(x, \alpha R)) \leq K \inf h(U(x, \alpha R)) \quad \text{for all } h \in \mathcal{H}^+(U(x, R)).$$

There are $\beta \in (0, 1)$ and $C \geq 1$ such that, for $x \in \mathbb{R}^d$ and $R > 0$,

$$|h(y) - h(x)| \leq C \|h\|_\infty \left(\frac{|x - y|}{R} \right)^\beta \quad \text{for all } y \in U(x, R), h \in \mathcal{H}_b(U(x, R)),$$

and every universally measurable function on \mathbb{R}^d , which is harmonic on an open set U in X_0 , is continuous on U .

REMARK 5.3. For a sufficient property which is weaker than (5.2) see (6.5).

6 Application based on an Ikeda-Watanabe estimate

To cover more general processes let us return to the setting of Section 4, where we have the following: A separable metric space (X, ρ) and harmonic measures μ_x^U on X , $x \in X$, U open sets in X , which satisfy (M_0) and (M_1) (see Section 2), and a Borel measurable function $G: X \times X \rightarrow (0, \infty]$ such that $(G_1) - (G_3)$ hold. In particular, we have an open set X_0 in X , balls $U(x, r) := \{y \in X: \rho(x, y) < r\}$, and $R_0(x) := \sup\{r > 0: \overline{U(x, r)} \subset X_0\}$, $x \in X_0$.

For every $V \in \mathcal{U}(X_0)$, let G_V be the associated (Green) function on V , that is,

$$G_V(x, y) := G(x, y) - \int G(z, y) d\mu_x^V(z), \quad x, y \in V,$$

and $G_V := 0$ outside $V \times V$. We suppose that we have the following relation between the functions $G_{U(x, r)}(x, \cdot)$ and the harmonic measures $\mu_x^{U(x, r)}$.

(IW) *There exist $\lambda \in \mathcal{M}(X)$, a kernel N on X , $M_{IW} \geq 1$ and $C_{IW} \geq 1$ such that, for all $x \in X_0$, $0 < r < R_0(x)$ and Borel sets E in $X \setminus \overline{U(x, M_{IW}r)}$,*

$$(6.1) \quad C_{IW}^{-1} \mu_x^{U(x, r)}(E) \leq \int G_{U(x, r)}(x, z) N(z, E) d\lambda(z) \leq C_{IW} \mu_x^{U(x, r)}(E).$$

REMARK 6.1. With $C_{IW} = 1$ and $M_{IW} = 1$, (6.1) is part of the Ikeda-Watanabe formula which holds for all (temporally homogeneous) Lévy processes (see [16, Example 1 and Theorem 1]).

We are indebted to a referee of the manuscript [14] (which merged into the present paper) for the hint that, in the Examples 2.1,2, the Ikeda-Watanabe formula always holds under mild duality assumptions (where λ is the Revuz measure of a positive continuous additive functional H given by a Lévy system (N, H) for a suitable Hunt process (X_t) and an excessive reference measure m associated with (X, \mathcal{W}) ; see [3, 6, 8, 25]):

$$\lambda(A) = \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}^m \int_0^t 1_A \circ X_s dH_s, \quad A \in \mathcal{B}(X).$$

We shall get the following results, where it only remains to prove that property (HJ) is satisfied (see (3.4) and Theorems 4.10 and 4.12).

THEOREM 6.2. *Suppose that there exist $C \geq 1$ and $\alpha \in (0, 1)$ such that, for all $x \in X_0$, $0 < r < \alpha R_0(x)$, $y, y' \in U(x, \alpha r)$,*

$$(6.2) \quad N(y', \cdot) \leq CN(y, \cdot) \quad \text{on } U(x, r)^c$$

and

$$(6.3) \quad \int_{U(x, \alpha r)} g(\rho(x, z)) d\lambda(z) \leq C \int_{U(y, 2\alpha r)} g(\rho(y, z)) d\lambda(z).$$

Then scaling invariant Harnack inequalities hold for functions in $\mathcal{H}^+(U(x, R))$, $x \in X_0$ and $0 < R < R_0(x)$.

Moreover, if $1 \in \mathcal{H}(X)$, then scaling invariant Hölder continuity holds for functions in $\mathcal{H}_b(U(x, R))$, $x \in X_0$ and $0 < R < R_0(x)$, and every universally measurable function on X , which is harmonic on an open set U in X_0 , is continuous on U .

Let us note that (6.3) trivially holds, if $X = \mathbb{R}^d$, $d \geq 1$, $\rho(x, y) = |x - y|$ and λ is Lebesgue measure.

THEOREM 6.3. *Suppose that $X = \mathbb{R}^d$, $\rho(x, y) = |x - y|$, the measure λ in (IW) is Lebesgue measure and there exists $c_2 \geq 1$ such that the normalized Lebesgue measure $\lambda_{U(x, r)}$ on $U(x, r)$ satisfies¹*

$$(6.4) \quad G\lambda_{U(x, r)} \leq c_2 g(r), \quad x \in X_0, 0 < r < R_0(x).$$

Moreover, assume that there are a measure $\tilde{\lambda}$ on \mathbb{R}^d , a Borel measurable function $n: \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ and $C \geq 1$, $\alpha \in (0, 1)$ such that $N(y, \cdot) = n(y, \cdot)\tilde{\lambda}$, $y \in X_0$, and, for all $y, y' \in X_0$ and $\tilde{z} \in X$,

$$(6.5) \quad n(y, \tilde{z}) \leq Cn(y', \tilde{z}), \quad \text{if } |y - \tilde{z}| \geq |y' - \tilde{z}| \quad \text{and} \quad |y - y'| < \alpha|y' - \tilde{z}|.$$

Then the conclusions of Theorem 6.2 prevail.

REMARK 6.4. *Suppose that there exists a function $n_0: [0, \infty) \rightarrow [0, \infty)$, $C_0 \geq 1$ and $\alpha \in (0, 1)$ such that $C_0^{-1}n_0(|x - y|) \leq n(x, y) \leq C_0n_0(|x - y|)$ and*

$$(6.6) \quad n_0(t) \leq C_0n_0(s), \quad \text{whenever } 0 < s < t < (1 + \alpha)s.$$

Then (6.5) is satisfied.

¹The inequality (6.4) trivially implies that (G_3) holds (see also [11, (1.14)]).

Thus rather general Lévy processes abundantly provide examples for our approach (see [9, 20, 22, 23, 24, 26]).

For the proofs of Theorem 6.2 and 6.3 we need the following simple statement, where $M := 2cc_Dc^2$ and $0 < \alpha_M < 1/4$ such that $Mg(r) \leq g(\alpha_M r)$, $r > 0$ (see (4.3)).

LEMMA 6.5. *Let $y \in X$, $r > 0$ and $0 < \alpha < \alpha_M$. Then*

$$(6.7) \quad G_{U(y,r)}(\cdot, y) \geq \frac{1}{2} G(\cdot, y) \quad \text{on } U(y, 2\alpha r).$$

Proof. Let $x \in U(y, 2\alpha r)$. Since $G(\cdot, y) \leq cg(r)$ on $U(y, r)^c$, we obtain that

$$\int G(z, y) d\mu_x^{U(y,r)} \leq cg(r) \|\mu_x^{U(y,r)}\| \leq (2cc_D)^{-1} g(\alpha r),$$

whereas $G(x, y) \geq c^{-1}g(2\alpha r) \geq (cc_D)^{-1}g(\alpha r)$. So (6.7) holds. \square

Proof of Theorem 6.2. To prove (HJ) we fix $0 < \alpha < (\alpha_M \wedge M_{IW}^{-1})/2$. Now let $x \in X_0$, $0 < r < \alpha R_0(x)$, $y \in U(x, \alpha^2 r)$, and let E be a Borel set in $U(x, r)^c$. Then E is contained in both $U(x, M_{IW}\alpha r)^c$ and $U(y, M_{IW}\alpha r)^c$. Hence, by (6.1) – (6.3), Lemma 6.5 and (2.3),

$$\begin{aligned} \mu_x^{U(x, \alpha^2 r)}(E) &\leq C_{IW} \int G_{U(x, \alpha^2 r)}(x, z) N(z, E) d\lambda(z) \\ &\leq cCC_{IW} N(y, E) \int_{U(x, \alpha^2 r)} g(\rho(x, z)) d\lambda(z) \\ &\leq cC^2 C_{IW} N(y, E) \int_{U(y, 2\alpha^2 r)} g(\rho(y, z)) d\lambda(z) \\ &\leq 2c^2 C^3 C_{IW} \int G_{U(y, \alpha r)}(y, z) N(z, E) d\lambda(z) \\ &\leq 2c^2 C^3 C_{IW}^2 \mu_y^{U(y, \alpha r)}(E) \leq 2c^2 C^3 C_{IW}^2 \mu_y^{U(x, r)}(E). \end{aligned}$$

Thus (HJ) holds (with α^2 in place of α). \square

Proof of Theorem 6.3 (cf. the proof of [9, Proposition 6]). First, we choose $0 < \alpha < (\alpha_M \wedge M_{IW}^{-1})/2$ such that (6.5) holds. Next, we fix $x \in X_0$, $0 < r < R_0(x)$ and a Borel set E in $U(x, r)^c$. Then $E \subset X \setminus \overline{U(y, M_{IW}\alpha r)}$ for every $y \in U(x, \alpha^2 r)$. By (6.1), (2.3) and Lemma 6.5,

$$\begin{aligned} \mu_x^{U(x, \alpha^2 r)}(E) &\leq cC_{IW} \int_{U(x, \alpha^2 r)} \int_E g(|x - z|) n(z, \tilde{z}) d\tilde{\lambda}(\tilde{z}) d\lambda(z), \\ \mu_y^{U(x, r)}(E) &\geq \mu_y^{U(y, \alpha r)}(E) \geq C_{IW}^{-1} \int G_{U(y, \alpha r)}(y, z) N(z, E) d\lambda(z) \\ &\geq (2C_{IW})^{-1} \int_{U(y, 2\alpha^2 r)} G(y, z) N(z, E) d\lambda(z) \\ &\geq (2cc_D C_{IW})^{-1} g(\alpha^2 r) \int_{U(x, \alpha^2 r)} \int_E n(z, \tilde{z}) d\tilde{\lambda}(\tilde{z}) d\lambda(z). \end{aligned}$$

Defining $\tilde{r} := \alpha^2 r$, we hence have to show that, with some constant $C' > 0$,

$$(6.8) \quad \int_{U(x, \tilde{r})} g(|x - z|) n(z, \tilde{z}) d\lambda(z) \leq C' g(\tilde{r}) \int_{U(x, \tilde{r})} n(z, \tilde{z}) d\lambda(z)$$

for every $\tilde{z} \in U(x, r)^c$. To that end let us fix $\tilde{z} \in U(x, r)^c$.

Let $B := U(x, \tilde{r}/2)$. Since $g(|x - y|) \leq g(\tilde{r}/2) \leq c_D g(\tilde{r})$ for every $y \in B^c$,

$$\int_{U(x, \tilde{r}) \setminus B} g(|x - y|) n(y, \tilde{z}) d\lambda(y) \leq c_D g(\tilde{r}) \int_{U(x, \tilde{r})} n(y, \tilde{z}) d\lambda(y).$$

Moreover, let

$$x' := x + \frac{3}{4} \frac{\tilde{z} - x}{|\tilde{z} - x|} \tilde{r} \quad \text{and} \quad B' := U(x', \tilde{r}/4),$$

so that $B' \subset U(x, \tilde{r}) \setminus B$. If $y \in B$ and $y' \in B'$, then $|y - \tilde{z}| \geq |y' - \tilde{z}|$ and $|y - y'| < 3\tilde{r}/2 < \alpha(r - \tilde{r}) < \alpha|y' - \tilde{z}|$, and therefore, by (6.5),

$$n(y, \tilde{z}) \leq \frac{C}{\lambda(B')} \int_{B'} n(y', \tilde{z}) d\lambda(y') = \frac{2^d C}{\lambda(B)} \int_{B'} n(y', \tilde{z}) d\lambda(y').$$

Hence

$$\begin{aligned} \int_B g(|x - y|) n(y, \tilde{z}) d\lambda(y) &\leq 2^d C \left(\int_{B'} n(y', \tilde{z}) d\lambda(y') \right) \cdot \left(\frac{1}{\lambda(B)} \int_B g(|x - y|) d\lambda(y) \right), \end{aligned}$$

where, by (6.4),

$$\frac{1}{\lambda(B)} \int_B g(|x - y|) d\lambda(y) \leq c_G \lambda_B(x) \leq c c_2 g(\tilde{r}/2) \leq c c_D c_2 g(\tilde{r}).$$

Thus (6.8) holds with $C' := c_D(1 + 2^d c c_2 C)$. \square

7 Intrinsic scaling invariant Harnack inequalities

In this section we shall weaken the assumptions and prove intrinsic scaling invariant Harnack inequalities, where the metric is derived from the Green function. We start with the same setting as in Section 4 (assuming that X_0 is a proper subset of X) and suppose that we have a Borel measurable function $G: X \times X \rightarrow (0, \infty]$ which satisfies (G_1) . Let us define

$$\begin{aligned} V(x, s) &:= \{y \in X: G(y, x)^{-1} < s\}, \quad x \in X, s > 0, \\ S_0(x) &:= \sup\{s > 0: \overline{V(x, s)} \subset X_0\}, \quad x \in X_0. \end{aligned}$$

Instead of (G_2) and (G_3) we assume the following properties (where also the case $w = 1$ is of interest).

(G'₂) For every $x \in X$, $G(x, x) = \lim_{y \rightarrow x} G(y, x) = \infty$, and there exists a Borel measurable function w on X , $0 < w \leq 1$, such that

$$(7.1) \quad \int w d\mu_x^U \leq w(x) \quad \text{for all open sets } U \text{ in } X \text{ and } x \in X,$$

and G has the (w, w) -triangle property, that is, for some $\tilde{c} > 1$, the function

$$\tilde{G}: (x, y) \mapsto \frac{G(x, y)}{w(x)w(y)}$$

satisfies

$$(7.2) \quad \tilde{G}(x, z) \wedge \tilde{G}(y, z) \leq \tilde{c}\tilde{G}(x, y), \quad x, y, z \in X.$$

Moreover, $\lambda := \inf w(X_0) > 0$ and, for every $x \in X$ and neighborhood V of x , the function $G(\cdot, x)/w$ is bounded on V^c .

(\overline{G}'_3) There exists $c_3 \geq 1$ such that, for all $x \in X_0$ and $0 < s < S_0(x)$,

$$(7.3) \quad \|\varepsilon_y^{\overline{V(x,s)}}\| \geq c_3^{-1}sG(y, x), \quad y \in V(x, s)^c.$$

Moreover, we introduce the following property.

(HJ') There exist $\alpha \in (0, 1)$ and $c_J \geq 1$ such that, for all $x \in X_0$ and $0 < s < S_0(x)$,

$$(7.4) \quad \mu_x^{\mathring{V}(x, \alpha s)} \leq c_J \mu_y^{\mathring{V}(x, s)} \quad \text{on } \mathring{V}(x, s)^c, \quad y \in \mathring{V}(x, \alpha^2 s).$$

REMARKS 7.1. 1. Clearly, (G_2) implies (G'_2) with $w = 1$, by Proposition 4.3.

2. In interesting cases, the function G does not have the $(1, 1)$ -triangle property, and hence (G_2) cannot hold. However, in these cases the (w, w) -triangle property frequently holds with $w = G(\cdot, y_0) \wedge 1$, where y_0 is some fixed point in X_0 . This is already the case in classical potential theory and the theory of Riesz potentials, if X is an open ball in \mathbb{R}^d .

PROPOSITION 7.2. There exist a metric $\tilde{\rho}$ for the topology of X , $\gamma \geq 1$ and $C \geq 1$ such that, for all $(x, y) \in X \times X$,

$$(7.5) \quad C^{-1}\tilde{\rho}(x, y)^{-\gamma} \leq \tilde{G}(x, y) \leq C\tilde{\rho}(x, y)^{-\gamma}.$$

In particular, (G_2) holds for \tilde{G} with $\tilde{g}(r) := r^{-\gamma}$. Moreover, let

$$\begin{aligned} \tilde{U}(x, r) &:= \{y \in X : \tilde{\rho}(x, y) < r\}, \quad x \in X, r > 0, \\ \tilde{R}_0(x) &:= \sup\{r > 0 : \overline{\tilde{U}(x, r)} \subset X_0\}, \end{aligned}$$

and $\beta := (\lambda/C)^{2/\gamma}$. Then, for all $x \in X_0$ and $r > 0$,

$$(7.6) \quad \tilde{U}(x, \beta r) \subset V(x, C^{-1}r^\gamma) \subset \tilde{U}(x, r).$$

Proof. Since $\tilde{G} = \infty$ on the diagonal, (7.2) implies that $\tilde{G}(y, x) \leq \tilde{c}\tilde{G}(x, y)$ and

$$(x, y) \mapsto \tilde{G}(x, y)^{-1} + \tilde{G}(y, x)^{-1}$$

defines a quasi-metric on X which is equivalent to \tilde{G}^{-1} . So, by [15, Proposition 14.5] (see also [11, Proposition 6.1]), there exist a metric $\tilde{\rho}$ on X , $\gamma \geq 1$ and $C \geq 1$ such that (7.5) holds.

Now let $x \in X$, $r > 0$ and $s := C^{-1}r^\gamma$. If $y \in \tilde{U}(x, \beta r)$, then

$$G(y, x) \geq \lambda^2 \tilde{G}(y, x) \geq \lambda^2 C^{-1} \tilde{\rho}(x, y)^{-\gamma} > \lambda^2 C^{-1} (\beta r)^{-\gamma} = s^{-1}.$$

Therefore $\tilde{U}(x, \beta r) \subset V(x, s)$. If $y \in V(x, s)$, then

$$C \tilde{\rho}(x, y)^{-\gamma} \geq \tilde{G}(y, x) \geq G(y, x) > Cr^{-\gamma},$$

and hence $\tilde{\rho}(x, y) < r$. So $V(x, s) \subset \tilde{U}(x, r)$, where $V(x, s)$ is a neighborhood of x , since $\lim_{z \rightarrow x} G(z, x) = \infty$.

Finally, if V is a neighborhood of x , there exists $M > 0$ with $G(\cdot, x)/w \leq M$ on V^c . Now let $r := (CM/w(x))^{-1/\gamma}$. Then, for every $y \in \tilde{U}(x, r)$,

$$G(y, x)/w(y) = w(x) \tilde{G}(y, x) \geq C^{-1} w(x) \tilde{\rho}(x, y)^{-\gamma} > M.$$

Hence $\tilde{U}(x, r) \subset V$. Thus $\tilde{\rho}$ is a metric for the topology of X . \square

We intend to prove the following theorem, where (HJ') trivially holds, if the measures μ_x^U are supported by the boundary of U .

THEOREM 7.3. *Suppose that (HJ') holds. Then there exist $\alpha \in (0, 1)$ and $K \geq 1$ such that, for all $x \in X_0$, $0 < R < \tilde{R}_0(x)$ and $h \in \mathcal{H}^+(\tilde{U}(x, R))$,*

$$(7.7) \quad \sup h(\tilde{U}(x, \alpha R)) \leq K \inf h(\tilde{U}(x, \alpha R)).$$

To that end we introduce normalized measures and normalized harmonic functions. For all $x \in X$, open sets U and closed sets A in X , let

$$(7.8) \quad \tilde{\mu}_x^U := \frac{w}{w(x)} \mu_x^U, \quad \tilde{\varepsilon}_x^A := \tilde{\mu}_x^{A^c} = \frac{w}{w(x)} \varepsilon_x^A.$$

For every $U \in \mathcal{U}(X_0)$, let $\tilde{\mathcal{H}}(U)$ be the set of all universally measurable real functions \tilde{h} on X such that, for all open sets V with $\overline{V} \subset U$ and $x \in V$, the function \tilde{h} is $\tilde{\mu}_x^V$ -integrable and

$$\int \tilde{h} d\tilde{\mu}_x^V = \tilde{h}(x).$$

Obviously,

$$(7.9) \quad \tilde{\mathcal{H}}(U) = \frac{1}{w} \mathcal{H}(U).$$

We shall prove that Theorem 4.10 holds for $\tilde{\mathcal{H}}^+(\tilde{U}(x, r))$. Then Theorem 7.3 follows easily using (7.9), since $w \leq 1$ and $w \geq \lambda$ on X_0 .

So let us verify the assumptions made for Theorem 4.10. It is immediately seen that (M_0) , (M_1) hold for the measures $\tilde{\mu}_x^U$ (the inequality $\|\tilde{\mu}_x^U\| \leq 1$ is equivalent to (7.1)). Further, for all closed sets $A \subset X_0$, measures ν on A and $x \in X$,

$$G\nu = w\tilde{G}(w\nu) \quad \text{and} \quad \frac{\lambda}{w(x)} \varepsilon_x^A \leq \tilde{\varepsilon}_x^A \leq \frac{1}{w(x)} \varepsilon_x^A.$$

Therefore (G_1) holds for the measures $\tilde{\varepsilon}_x^A$ and \tilde{G} with $\tilde{c}_1 := \lambda^{-1}c_1$ instead of c_1 (and measure $\tilde{\nu} := w\nu$ instead of ν). To deal with (G_3) we first show the following.

PROPOSITION 7.4. *Property (\bar{G}_3) holds for \tilde{G} , $\tilde{\rho}$ and $\tilde{g}(r) = r^{-\gamma}$.*

Proof. Let $\tilde{c}_3 := c_3C/\lambda^2$. Let $x \in X_0$, $0 < r < \tilde{R}_0(x)$ and $s := C^{-1}r^\gamma$. By Proposition 7.2, $V(x, s) \subset \tilde{U}(x, r)$. Let $y \in \tilde{U}(x, r)^c$. Using (7.3) we see that

$$\|\tilde{\varepsilon}_y^{\tilde{U}(x, r)}\| \geq \frac{\lambda}{w(y)} \|\varepsilon_y^{\overline{V(x, s)}}\| \geq c_3^{-1} \frac{\lambda}{w(y)} s G(y, x) \geq c_3^{-1} \lambda^2 s \tilde{G}(y, x) = \tilde{c}_3^{-1} r^\gamma \tilde{G}(y, x).$$

□

COROLLARY 7.5. *Property (G_3) holds for \tilde{G} , $\tilde{\rho}$, $\tilde{g}(r) = r^{-\gamma}$ and the capacity $\widetilde{\text{cap}}$ given by \tilde{G} .*

Proof. Propositions 7.4 and 4.6. □

LEMMA 7.6. *Suppose that (HJ') holds. Then (HJ) holds for $\tilde{\rho}$ and the measures $\tilde{\mu}_x^{\tilde{U}(x, r)}$, $x \in X_0$, $0 < r < \tilde{R}_0(x)$.*

Proof. Let $\tilde{\alpha} := \alpha\beta$, $x \in X_0$, $0 < r < \tilde{R}_0(x)$, $s := C^{-1}r^\gamma$. Then

$$(7.10) \quad \tilde{U}(x, \tilde{\alpha}^k r) \subset \overset{\circ}{V}(x, \alpha^k s), \quad k \in \mathbb{N}.$$

Indeed, $\tilde{\alpha}^k \leq \beta\alpha^k$, where $\alpha^{k\gamma} \leq \alpha^k$, since $\gamma \geq 1$. So (7.10) follows from the first inclusion in (7.6). Let $y \in \tilde{U}(x, \tilde{\alpha}^2 r)$. Then, by (2.3), (HJ') and (7.6),

$$\mu_x^{\tilde{U}(x, \tilde{\alpha}^2 r)} \leq \mu_x^{\overset{\circ}{V}(x, \alpha s)} \leq c_J \mu_y^{\overset{\circ}{V}(x, s)} \leq c_J \mu_y^{\tilde{U}(x, r)} \quad \text{on } \overset{\circ}{V}(x, s)^c,$$

which contains $\tilde{U}(x, r)^c$. Hence, by (7.8),

$$\tilde{\mu}_x^{\tilde{U}(x, \tilde{\alpha}^2 r)} = \frac{w}{w(x)} \mu_x^{\tilde{U}(x, \tilde{\alpha}^2 r)} \leq \lambda^{-1} c_J \frac{w}{w(y)} \mu_y^{\tilde{U}(x, r)} = \lambda^{-1} c_J \tilde{\mu}_y^{\tilde{U}(x, r)} \quad \text{on } \tilde{U}(x, r)^c.$$

□

As already indicated, Theorem 7.3 now follows by an application of Theorem 4.10 to $\tilde{\mathcal{H}}^+(\tilde{U}(x_0, R))$ and $\tilde{\rho}$ (recall the identity (7.9) and the inequalities $\lambda \leq w \leq 1$ on X_0).

References

- [1] R.F. Bass and D.A. Levin. Harnack inequalities for jump process. *Potential Anal.* 17: 375–388 (2002).
- [2] R.F. Bass and M. Kassmann. Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.* 354: 837–850 (2005).
- [3] A. Benveniste and J. Jacod. Systèmes de Lévy des processus de Markov. *Inventiones math.* 21: 183–198 (1973).
- [4] L. Beznea and N. Boboc. *Potential Theory and Right Processes*. Mathematics and its Applications 572, Kluwer Academic Publishers, Dordrecht, 2004.
- [5] J. Bliedtner and W. Hansen. *Potential Theory – An Analytic and Probabilistic Approach to Balayage*. Universitext. Springer, Berlin, 1986.
- [6] R.M. Blumenthal and R.K. Gettoor. *Markov Processes and Potential Theory*. Academic Press, New York and London, 1968.
- [7] K. Bogdan, A. Stós and P. Sztonyk. Potential theory for Lévy stable processes. *Bull. Pol. Acad. Sci. Math.* 50: 361–372 (2002).
- [8] S.E. Graversen and M. Rao. On strong Markov duals. *Math. Scand.* 49: 275–282 (1981).
- [9] T. Grzywny. On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes. *Potential Anal.* 41: 1–29 (2014).
- [10] W. Hansen. *Three views on potential theory*. A course at Charles University (Prague), Spring 2008. <http://www.karlin.mff.cuni.cz/~hansen/lecture/course-07012009.pdf>.
- [11] W. Hansen. *Liouville property, Wiener’s test and unavoidable sets for Hunt processes*. arXiv: 1409.7532v3.
- [12] W. Hansen. *Intrinsic Hölder continuity of harmonic functions*. arXiv: 1510.05800v2.
- [13] W. Hansen and I. Netuka. Representation of potentials. *Rev. Roumaine Math. Pures Appl.* 59: 93–104 (2014).
- [14] W. Hansen and I. Netuka. Harnack inequalities for Hunt processes with Green function. arXiv 1409.7532.
- [15] J. Heinonen. *Lectures on analysis on metric spaces*. Springer, New York, 2001.
- [16] N. Ikeda and S. Watanabe. On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. *J. Math. Kyoto Univ.* 2: 79–95 (1962).
- [17] M. Kassmann. Harnack inequalities: an introduction. *Bound. Value Probl.* 2007, Art. ID 81415, 21 pp.

- [18] M. Kassmann and A. Mimica. Analysis of jump processes with non-degenerate jumping kernel. *Stoch. Process. Appl.* 123: 629–650 (2013).
- [19] M. Kassmann and A. Mimica. Intrinsic scaling properties for nonlocal operators. arXiv:1412.7566v3. To appear in *J. Eur. Math. Soc.*
- [20] P. Kim and A. Mimica. Harnack inequalities for subordinate Brownian motions. *Electron. J. Probab.* 17: no. 37, 23 pp. (2012).
- [21] A. Mimica. Harnack inequalities for some Lévy processes. *Potential Anal.* 32: 275–303 (2010).
- [22] A. Mimica. Harnack inequality and Hölder regularity estimates for a Lévy process with small jumps of high intensity. *J. Theor. Probab.* 26: 329–348 (2013).
- [23] A. Mimica. On harmonic functions of symmetric Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50: 214–235 (2014).
- [24] M. Rao, R. Song, and Z. Vondraček. Green function estimates and Harnack inequality for subordinate Brownian motions. *Potential Anal.* 25: 1–27 (2006).
- [25] D. Revuz. Mesures associées aux fonctionnelles additives de Markov. I. *Trans. Amer. Math. Soc.* 148: 501–531 (1970).
- [26] H. Šikić, R. Song, and Z. Vondraček. Potential theory of geometric stable processes. *Probab. Theory Related Fields* 135: 547–575 (2006).
- [27] R. Song and Z. Vondraček. Harnack inequality for some classes of Markov processes. *Math. Z.* 246: 177–202 (2004).
- [28] R. Song and Z. Vondraček. Harnack inequality for some discontinuous Markov processes with a diffusion part. *Glas. Mat.* 40: 177–187 (2005).
- [29] Z. Vondraček. Basic potential theory of certain nonsymmetric strictly α -stable processes. *Glas. Mat.* 37: 193–215 (2002).

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